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TITLE: LOW-DIMENSIONAL BEHAVIOR OF THE PATTERN FORMATION CAHN-HILLIARD EQUATION

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LOW-DIMENSIONAL BEHAVIOR OF THE PATTERN FORMATION CAHN-HILLIARD EQUATION

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We investigate the fourth-order Cahn-Hilliard parabolic partial differential equation which describes pattern formation in phase transition. Neumann and periodic boundary conditions are considered for a domain in R^n, $1 \leq n \leq 3$. This equation is characterized by a negative (backward) second order diffusion and multiple steady states for the appropriate range of parameters. We establish compactness of the orbits in $H^1(\Omega)$ and convergence to some steady state. We demonstrate that the Cahn-Hilliard equation admits an intrinsic low dimensional behavior: in R^1, the number of determining modes (in a Galerkin expansion) is proportional to $L^{3/2}$; where L, the diameter of the domain, is also proportional to the number of unstable modes for the linearized equation. Similar results hold for n = 2,3.

1. INTRODUCTION

We investigate the low dimensional behavior of the Cahn-Hilliard equation with a quartic homogeneous free energy, in R^n, $1 \leq n \leq 3$:

$$\frac{\partial u}{\partial t} = \operatorname{div}\,[M(u)\,\nabla\,(-\Delta u + \alpha u^3 - \beta u)]$$

$$\equiv \operatorname{div}\,[M(u)\,\nabla\,J(u)] \quad \text{in } \Omega \subset R^n \quad ,$$

$$u(0) = u_0 \in H^2(\Omega) \quad , \alpha > 0 \quad \text{and} \quad \beta > 0 \quad ; \tag{1.1a}$$

the following hypotheses are made for the mobility coefficient M(u):

$$M(u) > 0 \quad , \text{monotone non-increasing in } |u|,\ C^1$$

$$\text{and } M(u) > M(0)\ \exp -\lambda|u| \quad , \lambda > 0 \quad ; \tag{1.1b}$$

the boundary conditions on $\partial\Omega$ (boundary of the pattern cell) are either of the Neumann type or periodic (periodic cell structure):

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0 \quad , \left.\frac{\partial J}{\partial \nu}\right|_{\partial\Omega} = 0 \quad , \tag{1.1c}$$

or

$$u(x + Le_i t) = u(x,t) \quad 1 \leq i \leq n \quad , \tag{1.1d}$$

L being the size of a typical pattern cell.

Eq. (1.1) is in fact a normalized form for the classical Cahn-Hilliard equation [2,5,9]:

$$\frac{\partial c}{\partial t} = \operatorname{div}\,[M(u)\,\nabla\,(-\Delta c + b_2 c + b_3 c^2 + b_4 c^3)] \quad ,$$

$$b_2 \text{ either} > 0 \text{ or} < 0, \quad b_3 < 0, \; b_4 > 0 \quad , \tag{1.2}$$

with the same boundary conditions. As shown below (1.2) reduces to (1.1) by a simple translation $c(x,t) = u(x,t) + c^*$, c^* constant.

Eq. (1.2) is a continuum model for pattern formation resulting from phase transition. It is associated to a classical Landau-Ginzburg free energy [1]:

$$\hat{F} = \int_\Omega (\tfrac{1}{2}(\nabla\hat{c})^2 + f(\hat{c}))\,dx \quad , \quad \int_\Omega \hat{c}\,dx \equiv \int_\Omega c(x,0)\,dx = ct \quad , \tag{1.3a}$$

where the homogeneous free energy $f(c)$ is a quartic polynomial whose derivative is:

$$\frac{\partial f}{\partial c} = b_2 c + b_3 c^2 + b_4 c^3 \quad , \; b_3 < 0 \quad , \; b_4 > 0 \quad . \tag{1.3b}$$

Steady-state solutions of (1.2) are given by critical points of the <u>non-convex</u> functional F. The corresponding Euler-Lagrange equation is:

$$-\Delta\hat{c} + b_2\hat{c} + b_3\hat{c}^2 + b_4\hat{c}^3 = ct \quad , \tag{1.3c}$$

plus appropriate boundary conditions.

The influence of the homogeneous free energy function $f(c)$ appears in the sign of b_2 and the parameter B [9]:

$$B = -\frac{b_3}{(|b_2|b_4)^{\frac{1}{2}}} \quad . \tag{1.4}$$

If $b_2 \leq 0$, there is a "negative viscosity" destabilizing mechanism somewhat similar to the one observed in the Kuramoto-Sivashinsky equation for unstable flame fronts [6-8]. The <u>zero solution is unstable</u> and this regime is referred to as "unstable subspinodal." The special limit case $b_2 = 0$ is called the "spinodal regime."

If $b_2 > 0$ and $B^2 > 3$, the cubic $\frac{\partial f}{\partial c}$ defined in (1.3b) possesses two distinct extrema. If $B^2 < 3$, $b_2 > 0$, it is well known that zero is a monotonically stable attractor [5,9]. A. Novick-Cohen and L. A. Segel [9] have extensively studied the case $3 \leq B^2 < \infty$ in a one-dimensional geometry. They have specified the full set of equilibrium solutions. They have also established that for $4.5 \leq B^2 < \infty$, the basin of attraction of zero is bounded, whereas there exists at least another nontrivial equilibrium with its own basin of attraction. $B^2 = 4.5$ is the distinguished "binodal" case.

We investigate some global dynamical properties of (1.2) when $b_2 > 0$ and $B^2 > 3$, or $b_2 \leq 0$. <u>Either case reduce to the normalized equation</u> (1.1); set:

$$u(x,t) = c(x,t) - c^* \quad , \tag{1.5a}$$

where

$$c^* = -b_3/3b_4 > 0 \quad , \tag{1.5b}$$

and is such that

$$\left.\frac{\partial^3 f}{\partial c^3}\right|_{c=c^*} = 0 \quad ;$$

through the translation (1.5), the cubic $\frac{\partial f}{\partial c}$ is changed into:

$$\frac{\partial f}{\partial c} = c^* + [b_2 - \frac{1}{3}\frac{b_3^2}{b_4}]\, u + b_4\, u^3 \quad . \tag{1.6a}$$

We define

$$\alpha = b_4 > 0 \tag{1.6b}$$

$$\beta = -\,[b_2 - \frac{1}{3}\frac{b_3^2}{b_4}] \quad , \beta > 0 \quad ; \tag{1.6c}$$

indeed $B^2 > 3$, $b_2 > 0$ implies $\beta > 0$. Injecting (1.5) and (1.6) into the Cahn-Hilliard Eq. (1.2) yields the normalized form (1.1), with $M \equiv M(c^* + u)$, and $u_0 = c(x,0) - c^*$.

In Section 1, we verify boundedness of orbits in $H^1(\Omega)$ and the existence of Lyapunov functional. Although the above is implicit in the literature, compactness of orbits in $H^1(\Omega)$ has not previously been established, to our knowledge. This is done in Section 2, and enables the correct application of a classical topological dynamics theorem of Hale [4]: all orbits strongly converge in $H^1(\Omega)$ to critical points of the non-convex functional (1.3a).

However, the most important results are found in Section 4; we establish the intrinsically low-dimensional behavior of the Cahn-Hillard equation. Essentially, we project any orbit onto the linear manifold of the first m-eigenmodes of the biharmonic Δ^2. Suppose that the m-dimensional projected orbit converges to some m-dimensional fixed point; we will say that the first m-eigenmodes are determining if this implies convergence of the infinite dimensional orbit.

Following ideas developed in the Navier-Stokes context by Foias-Manley-Temam-Treve [3], we prove that for the one-dimensional Cahn-Hilliard equation:

$$m \geq ct\ L^{3/2} \quad ,$$

where L is the pattern size.

L is also proportional to the number of unstable modes of (1.1) linearized at $u = 0$; indeed the eigenvalue spectrum is:

$$\Lambda_k = \beta^2\ \left(-\left(\frac{2\pi k}{\sqrt{\beta}L}\right)^4 + \left(\frac{2\pi k}{\sqrt{\beta}L}\right)^2\right) \quad , k = 0,1,2,\ldots.$$

and

$$\# \{\Lambda_k | \Lambda_k > 0\} = [\frac{\sqrt{\beta}}{2\pi} L] \quad ,$$

where [a] is the usual integer part of a. So for the determining modes:

$$m \geq ct \; (\# \text{ unstable modes})^{3/2} \quad ;$$

in some heuristic sense, the impact of the nonlinearity is reflected only through the exponent ½. Similar results hold for n = 2 and n = 3, periodic boundary conditions.

To simplify the technical derivations, we restrict ourselves to $M(u) = \text{constant}$; the general case is easily disposed of, as soon as one obtains an estimate such as:

$$\overline{\lim_{t\to\infty}} ||u(x,t)||_{L^\infty(\Omega)} < K \; ;$$

then from (1.1b)

$$0 < M(0) \leq M(u) \leq M(K) \quad .$$

2. BOUNDEDNESS OF ORBITS IN $H^1(\Omega)$: THE LYAPUNOV FUNCTION

We consider the normalized problem:

$$\frac{\partial u}{\partial t} - \Delta J(u) = 0 \text{ in } \Omega \quad , \tag{2.1a}$$

$$J(u) = -\Delta u + \alpha u^3 - \beta u \quad , \alpha \text{ and } \beta > 0$$

$$u(0) = u_0 \in H^2(\Omega) \tag{2.1b}$$

with either

- periodic boundary conditions , $u(x + Le_i, t) = u(x,t)$, $1 \leq i \leq n$ (2.1c)

(L being the size of a typical pattern cell) or

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = \left.\frac{\partial J}{\partial \nu}\right|_{\partial\Omega} = 0 \quad . \tag{2.1d}$$

In this section, $\Omega \subset R^n$, $1 \leq n \leq 3$.

First we have the:

__Lemma 2.1__. *$\bar{u}(t) \equiv \bar{u}(0)$, where $\bar{u}(t)$ is the average $\frac{1}{|\Omega|} \int u(x,t)\, dx$ and $|\Omega| = \text{meas } \Omega$.*

Remark 2.2. The previous lemma implies that Poincaré-like inequalities hold, as u can be renormalized to a function of null mean value. From now on, we set

$$||u|| = (\int u^2\, dx)^{1/2} \quad ,$$

unless specified otherwise.

We now look for a Lyapunov function associated with (2.1). Multiply (4.1) by J(u) and integrate by parts over Ω. With either set of boundary conditions:

$$\int_\Omega \frac{\partial u}{\partial t} J(u)\, dx + \int_\Omega (\nabla J(u))^2\, dx = 0 \tag{2.2a}$$

and injecting the explicit form of J(u) into the first integral:

$$\frac{d}{dt}\left(\frac{1}{2} \int_\Omega (\nabla u)^2\, dx - \frac{\beta}{2} \int_\Omega u^2 dx + \frac{\alpha}{4} \int_\Omega u^4 dx \right) + \int (\nabla J)^2 dx = 0 \quad . \tag{2.2b}$$

Let us define V(t) as:

$$V(t) = \frac{1}{2} \int_\Omega (\nabla u)^2\, dx - \frac{\beta}{2} \int_\Omega u^2 dx + \frac{\alpha}{4} \int_\Omega u^4 dx \quad . \tag{2.3}$$

Then (2.2b) implies:

$$\frac{d}{dt} V(t) \le 0 \quad . \tag{2.4}$$

To establish that V(t) is a Lyapunov function, we must show the boundedness of orbits in $H^1(\Omega)$ and that V(t) is <u>bounded</u> <u>from</u> <u>below</u> in $H^1(\Omega)$. Remark that:

$$V(t) = \frac{1}{2} \int_\Omega (\nabla u)^2 dx + \int_\Omega \left(\frac{\sqrt{\alpha}}{2} u^2 - \frac{\beta}{2\sqrt{\alpha}}\right)^2 dx - \frac{\beta^2}{4\alpha} |\Omega| \quad ; \tag{2.5}$$

now

$$V(t) \le V(0) \quad , \tag{2.6}$$

so

$$\frac{1}{2} \int_\Omega (\nabla u)^2\, dx + \int_\Omega \left(\frac{\sqrt{\alpha}}{2} u^2 - \frac{\beta}{2\sqrt{\alpha}}\right)^2 dx \le \frac{1}{2} \int_\Omega (\nabla u_0)^2\, dx + \int_\Omega \left(\frac{\sqrt{\alpha}}{2} u_0^2 - \frac{\beta}{2\sqrt{\alpha}}\right)^2 dx \quad . \tag{2.7}$$

This proves the

Theorem 2.3. $\overline{\lim}_{t\to\infty} ||\nabla u(t)|| \le F(u_0)$, *where*

$$F(u_0) = (||\nabla u_0^2|| + 2 \int_\Omega (\frac{\sqrt{\alpha}}{2} u_0^2 - \frac{\beta}{2\sqrt{\alpha}})^2 \, dx)^{\frac{1}{2}} \quad . \tag{2.8}$$

Corollary 2.4. $\overline{\lim\limits_{t\to\infty}} ||u||_{L^4}$ *is bounded.*

Proof. Use the continuous imbedding

$$H^1(\Omega) \hookrightarrow L^4(\Omega) \quad , \; n \leq 4$$

or specifically Eq. (2.7), together with Poincaré's inequality.

Corollary 2.5. $V(t)$ is a continuous, bounded from below, Lyapunov functional on $H^1(\Omega)$.

Remark 2.6. All of the above results are valid if we consider the more general equation (1.1) with the coefficient of diffusion $M(u)$ given as in (1.1b). Indeed:

$$\frac{\partial u}{\partial t} - \operatorname{div} M(u) \nabla J(u) = 0 \quad ;$$

multiplying by $J(u)$ and integrating over Ω:

$$\int_\Omega \frac{\partial u}{\partial t} J(u) \, dx + \int_\Omega M(u) \; (\nabla J)^2 \, dx = 0 \quad ,$$

and we still have

$$\frac{d}{dt} V(t) \leq 0 \quad ,$$

with $V(t)$ same as in (2.3).

3. ASYMPTOTIC BEHAVIOR OF ORBITS.

We wish to establish some kind of convergence of the orbits $u(x,t)$ to the critical manifold M of fixed points $\hat{u}(x)$ of:

$$-\Delta\hat{u} + \alpha \, \hat{u}^3 - \beta \, \hat{u} = \gamma \tag{3.1a}$$

$$\int_\Omega \hat{u} \, dx = |\Omega|\bar{u}(0) \tag{3.1b}$$

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \text{ or periodic boundary conditions} \quad . \tag{3.1c}$$

To apply classical topological dynamics results of Hale [4], we first need the relative compactness of orbits $u(t)$ in $H^1(\Omega)$:

Theorem 3.1. $\overline{\lim\limits_{t\to\infty}} \|D^2u\|$ *is bounded*[1], *for either periodic boundary conditions (2.1c) or Neumann conditions (2.1d) if* $\Omega \subset R^1$; *and for periodic boundary conditions if* $\Omega \subset R^2$ *or* R^3.

The proof is technical and will be outlined below. Theorem 3.1 ensures the relative compactness of the orbit $u(t)$ in $H^1(\Omega)$; hence, the ω-limit set associated to u_0 is nonempty, compact, invariant and connected. Using a classical theorem for such flows with Lyapunov functions [4], namely that $V(t)$ is constant on $\omega(u_0)$, we deduce:

Corollary 3.2. As $t \to \infty$, $\lim$ dist $|u(x,t) - M| = 0$ in $H^1(\Omega)$, *for either boundary conditions if* $\Omega \subset R^1$, *and for periodic boundary conditions if* $\Omega \subset R^2$ *or* R^3.

Remark 3.3. Problem (3.1) usually admits multiple solutions, whether one considers β or $L = \text{diam}\,\Omega$ as a bifurcation parameter [9].

Proof of Theorem 3.1. Multiply (2.1) by $\dfrac{\partial^4}{\partial x_1^{2\delta_1} \cdots \partial x_n^{2\delta_n}}\, u$, integrate by parts and take the sumation over all $\delta = (\delta_1, \ldots, \delta_n)$ such that $|\delta| = 2$; we get:

$$\frac{1}{2}\frac{d}{dt}\|D^2u\|^2 + \|D^4u\|^2 - \beta\|D^3u\|^2 = \sum_{|\delta|=2} \alpha \int \Delta u^3\, D^{2\delta}u\, dx$$

$$= \sum_{|\delta|=2} \left(6\alpha \int u|\nabla u|^2 D^{2\delta}u\, dx + 3\alpha \int u^2\, \Delta u\, D^{2\delta}u\, dx\right) \quad . \tag{3.2}$$

Apply Cauchy-Schwartz and Cauchy-Young's inequalities to the R.H.S. of (3.2):

$$\frac{1}{2}\frac{d}{dt}\|D^2u\|^2 + (1-\varepsilon)\|D^4u\|^2 \leq \beta\|D^3u\|^2 + C(\varepsilon)\int u^2(\nabla u)^4\, dx$$

$$+ C(\varepsilon)\int u^4\, (\Delta u)^2\, dx \quad ; \tag{3.3}$$

from now on $C(\varepsilon)$ will be a generic symbol for any constant depending upon ε.

We will estimate:

$$J_1 = \int u^2\, (\nabla u)^4\, dx \quad , \tag{3.4}$$

$$J_2 = \int u^4\, (\Delta u)^2\, dx \quad . \tag{3.5}$$

---

(1) For brevity, we set $\|D^k u\|^2 = \sum\limits_{|\alpha|=k} \|D^\alpha u\|^2$.

We will need the Agmon inequalities (for functions periodic and/or with zero mean value):

$$||u(t)||_{L^\infty} \le \begin{cases} \gamma_1 ||u(t)||^{\frac{1}{2}} \, ||\nabla u(t)||^{\frac{1}{2}} \quad , \text{ if } n = 1 \quad , \\ \gamma_2 ||u(t)||^{\frac{1}{2}} ||\Delta u(t)||^{\frac{1}{2}} \quad , \text{ if } n = 2 \\ \gamma_3 ||u(t)||^{\frac{1}{4}} \, ||\Delta u(t)||^{\frac{3}{4}} \quad , \quad \text{if } n = 3 \quad . \end{cases} \tag{3.6}$$

We also need the following general interpolation inequalities:

$$||D^{k+1}u|| \le ||D^{k-1}u||^{1/3} \, |D^{k+2}u||^{2/3} \tag{3.7}$$

$$||D^k u|| \le ||D^{k-1}u||^{\frac{1}{2}} ||D^{k+1}u||^{\frac{1}{2}} \tag{3.8}$$

Also, as $H^{\frac{1}{2}} \hookrightarrow L^4$ $(n = 2)$ or $H^{\frac{3}{4}} \hookrightarrow L^4$ $(n = 3)$, we will need:

$$||Du||^4_{L^4} \le ||Du||^3 \, ||D^3u|| \quad , n = 2 \quad ; \tag{3.9a}$$

$$||Du||^4_{L^4} \le ||Du||^{5/2} \, ||D^3u||^{3/2} \, , n = 3 \quad ; \tag{3.9b}$$

which are obtained by interpolation of $H^{\frac{1}{2}}$ (resp. $H^{\frac{3}{4}}$) between L^2 and H^2. We will give explicit technical details only for $n = 2$. The case $n = 1$ and $n = 3$ are similar.

In (3.3), we first consider the term $\beta||D^3u||^2$; from (3.7) and using Cauchy-Young's inequality with $p = 3/2$, $q = 3$:

$$\begin{aligned} ||D^3u||^2 \le ||D^4u||^{4/3} \, ||Du||^{2/3} \, || &\le \varepsilon \, ||D^4u||^2 + C(\varepsilon) \, ||Du||^2 \\ &\le \varepsilon \, ||D^4u||^2 + C(\varepsilon) \quad , \end{aligned} \tag{3.10}$$

since $\overline{\lim_{t\to\infty}} ||\nabla u|| \le F(u_0)$ (Theorem 2.3).

Now estimate J_1 in (3.4):

$$\int u^2(\nabla u)^4 dx < ||u||^2_{L^\infty} \, ||\nabla u||^4_{L^4} \quad ;$$

using Agmon's inequalities (3.6) and the interpolation inequality (3.9a):

$$J_1 < Ct \, ||u|| \, ||D^2u|| \, ||Du||^3 \, ||D^3u|| \quad ,$$

and from Theorem 2.3:

$$J_1 < Ct \, ||D^2u|| \, ||D^3u|| < Ct \, ||D^3u||^2$$

(using Poincaré's inequality) and

$$J_1 < \varepsilon \, ||D^4u||^2 + C(\varepsilon) \quad , \tag{3.11}$$

following (3.10).

Now estimate J_2 in (3.5):

$$\int u^4 (\Delta u)^2 \, dx \leq ||\Delta u||^2_{L^\infty} ||u^4||^4_{L^4} \quad ;$$

using Agmon's inequalities (3.6):

$$J_2 \leq Ct \; ||\Delta u|| \; ||D^4 u|| \; ||u^4||^4 \leq Ct \; ||D^2 u|| \; ||D^4 u|| \quad ,$$

(using Corollary 2.4); now using the interpolation inequality (3.8):

$$J_2 \leq Ct \; ||Du||^{\frac{1}{2}} \; ||D^3 u||^{\frac{1}{2}} \; ||D^4 u|| \leq Ct \; ||D^3 u||^{\frac{1}{2}} \; ||D^4 u|| \quad ;$$

but from the interpolation inequality (3.7):

$$||D^3 u|| \leq ||Du||^{1/3} \; ||D^4 u||^{2/3} \quad ;$$

so:

$$J_2 \leq Ct \; ||Du||^{1/6} \; ||D^4 u||^{4/3} \quad ,$$

and using Cauchy-Young's inequality with $p = 3/2$, $q = 3$:

$$J_2 \leq \varepsilon \; ||D^4 u||^2 + C(\varepsilon) \; ||Du||^{\frac{1}{2}} \quad ,$$

$$J_2 \leq \varepsilon \; ||D^4 u||^2 + C(\varepsilon) \quad . \tag{3.12}$$

We now collect all terms in Eq. (3.3), applying (3.10, 3.11, 3.12):

$$\frac{1}{2} \frac{d}{dt} ||D^2 u||^2 + (1 - 3\varepsilon - \beta\varepsilon) \; ||D^4 u||^2 < C(\varepsilon) \quad . \tag{3.13}$$

We conclude with the help of Poincaré's inequality and Gronwall's Lemma, that:

$$\overline{\lim_{t \to \infty}} \; ||D^2 u|| \; < \infty \quad . \qquad \square$$

4. NUMBER OF DETERMINING MODES

This section gives our main result, namely an upper bound of the number of determining modes for any solution of the Cahn-Hilliard equation (2.1) with periodic boundary conditions. This bound is formulated in terms of L. Although we give the detailed derivation for space dimension $n = 1$, analogue results can easily be derived for $n = 2$ and $n = 3$.

Consider u,v two solutions of (2.1), corresponding to two initial data (in $H^2(\Omega)$); set $w = u-v$. Due to the periodicity of u,v, we can use a Fourier mode decomposition of w and set:

$$P_m w(x,t) = \sum_{|k| \leq m} w_k(t) \exp \frac{2i\pi}{L} k.x \tag{4.1}$$

where $k \in Z^n$, and $w_k(t)$ is the k^{th} Fourier coefficient of $w(x,t)$. We will also use:

$$Q_m \, w(x,t) = (I - P_m) w(x,t) \quad . \tag{4.2}$$

Definition 4.1. *We say that the first m Fourier modes of $w = u-v$ are determining if:*

$$\lim_{t\to\infty} ||P_m (u(t) - v(t))|| = 0 \rightarrow \lim_{t\to\infty} ||u(t) - v(t)|| = 0 \quad . \tag{4.3a}$$

Remark 4.2. For Neumann boundary conditions (2.1d), we use the appropriate eigenfunctions of (Δ^2) as a Galerkin basis in (4.1 - 4.2).

Remark 4.3. If Ξ is a compact positive invariant set under the semi-flow defined in Section 3, then from (4.3) we deduce:

$$\lim_{t\to\infty} \mathrm{dist} =(P_m u(t), P_m \Xi) = 0 \rightarrow \lim_{t\to\infty} \mathrm{dist}(u(t), \Xi) = 0 \quad ,$$

since $v(t) \in \Xi$ for all times if $v(0) \in \Xi$.

In particular, if $u \equiv u^*$, where u^* is some equilibrium solution belonging to the set of M of fixed points (cf. Eq. (3.1), then:

$$\lim_{t\to\infty} ||P_m u(t) - P_m u^*|| = 0 \rightarrow \lim_{t\to\infty} ||u(t) - u^*|| = 0 \quad ; \tag{4.3b}$$

if the projection of the orbit converges to some (projected) fixed point, the same is true of the infinite-dimensional orbit.

The main result of this section is stated for space dimension $n = 1$; with $\Omega = [0,L]$ and periodic boundary conditions:

Theorem 4.4. *The first m Fourier modes are determining if*

$$m + 1 \geq K L^{3/2} \quad , \tag{4.4}$$

where K is some constant depending on α,β and ζ_0, with initial values $||\nabla u(0)|| \leq \zeta_0$.

Proof of Theorem 4.4. For sake of brevity, in the sequel, we will denote $q_m \equiv Q_m w$, $p_m \equiv P_m w$. Now, if u,v are two solutions, w satisfy the following equation:

$$\frac{\partial w}{\partial t} + \Delta(\Delta w + \beta w - \alpha[u^2 + uv + v^2]w) = 0 \quad . \tag{4.5a}$$

Multiplying by q_m and integrating:

$$\frac{1}{2} \frac{d}{dt} ||q_m||^2 + ||\Delta q_m||^2 - \beta ||\nabla q_m||^2 - \alpha \int [u^2 + uv + v^2] w \Delta q_m dx = 0 \tag{4.5b}$$

But $w = q_m + p_m$, and so by Hölder's inequality:

$$\int (u^2 + uv + v^2)w \Delta q_m dx$$

$$\leq ||u^2 + uv + v^2||_{L^\infty} (||p_m|| + ||q_m||) ||\Delta q_m|| \tag{4.6}$$

and

$$\frac{1}{2}\frac{d}{dt}||q_m||^2 + \frac{1}{||q_m||^2}\{||\Delta q_m||^2 - \beta\,||\nabla q_m||^2$$

$$- \alpha||u^2 + uv + u^2||_{L^\infty}\,||\Delta q_m||\,||q_m||\}\,||q_m||^2$$

$$\leq \alpha\,||u^2 + uv + v^2||_{L^\infty}\,||\Delta q_m||\,||p_m|| \quad . \tag{4.7}$$

<u>We</u> <u>must</u> <u>prove</u> <u>that</u> $||p_m|| \to 0$ <u>implies</u> $||q_m|| \to 0$. This will be completed by verifying the three assumptions of the generalized Gronwall's Lemma 4.1 of [3]. We recall this Lemma:

Let $\xi(t)$ be an absolutely continuous nonnegative function on $(0,\infty)$ such that

$$\frac{d\xi}{dt} + A(t)\xi \leq B(t) \quad \text{a.e. on } (0,\infty) \quad ,$$

where $A(t)$ is a locally integrable function on $(0,\infty)$ satisfying for some T, $0 < T < \infty$:

$$\liminf_{t\to\infty} \int_t^{t+T} A\, ds = \gamma > 0 \tag{H1}$$

$$\limsup_{t\to\infty} \int_t^{t+T} A^-\, ds = \Gamma < \infty \quad , \tag{H2}$$

where $A^- = \max(-A, 0)$ and $B(t)$ is a measurable function on $(0,\infty)$ such that

$$B(t) \to 0 \quad , \quad t \to \infty \quad , \tag{H3}$$

then

$$\xi(t) \to 0 \text{ as } t \to \infty \quad .$$

(Here, we set $\xi(t) \equiv ||q_m(t)||^2$.) We define:

$$A_m(t) = 2\,\frac{||\Delta q_m|| - \beta\,||\nabla q_m||^2}{||q_m||^2} - 2\alpha\,\frac{||u^2 + uv + v^2||_{L^\infty}}{||q_m||}\,||\Delta q_m|| \tag{4.8}$$

$$B_m(t) = 2\alpha\,||u^2 + uv + v^2||_{L^\infty}\,||\Delta q_m||\,||p_m|| \quad , \tag{4.9}$$

$$\rho_m(t) = \frac{||\Delta q_m||^2}{||q_m||^2} \quad , \quad \tilde{\rho}_m(t) = \frac{1}{T}\int_t^{t+T} \rho_m(s)\, ds \quad , \tag{4.10}$$

$$R(u,v) = \alpha||u^2 + uv + v^2||_{L^\infty} \quad . \tag{4.11}$$

Inequality (4.7) now can be rewritten in a more compact way:

$$\frac{d}{dt} ||q_m||^2 + A_m(t)\ ||q_m||^2 \leq B_m(t) \quad . \tag{4.12}$$

We first verify Hypothesis (H1) from the generalized Gronwall's Lemma:

$$A_m(t) \geq \frac{2||\Delta q_m||^2}{||q_m||^2} - \frac{2\beta||\Delta q_m||}{||q_m||} - 2\ R(u,v)\ \frac{||\Delta q_m||}{||q_m||}$$

$$= 2\ \rho_m(t) - 2\ \beta\ \rho_m(t)^{\frac{1}{2}} - 2\ R(u,v)\ \rho_m(t)^{\frac{1}{2}} \quad . \tag{4.13}$$

From (4.13):

$$\frac{1}{T}\int_t^{t+T} A_m(s)\ ds \geq 2\ \tilde{\rho}_m(t) - 2\ \beta\ \tilde{\rho}_m(t)^{\frac{1}{2}} - \frac{2}{T}\int_t^{t+T} R(u,v)\ \rho_m(s)^{\frac{1}{2}}\ ds$$

$$\geq 2\ \tilde{\rho}_m(t) - 2\ \beta\ \tilde{\rho}_m(t)^{\frac{1}{2}} - 2\left(\frac{1}{T}\int_t^{t+T} R(u,v)^2\ ds\right)^{\frac{1}{2}} \tilde{\rho}_m(t)^{\frac{1}{2}}$$

$$= 2\ \tilde{\rho}_m(t)^{\frac{1}{2}} [\tilde{\rho}_m(t)^{\frac{1}{2}} - \beta - (\frac{1}{T}\int_t^{t+T} R(u,v)^2\ ds)^{\frac{1}{2}}] \quad , \tag{4.14}$$

where we use a classical interpolation inequality for $||\nabla q_m||^2$ and Jenssen's inequality. From (4.14), a sufficient condition for (H1) is:

$$\tilde{\rho}_m(t)^{\frac{1}{2}} \geq \beta + (\frac{1}{T}\int_t^{t+T} R(u,v)^2\ ds)^{\frac{1}{2}} \quad ; \tag{4.15}$$

but

$$\tilde{\rho}_m(t) \geq E_{m+1} \quad , \tag{4.16}$$

where E_{m+1} is the $(m+1)^{th}$ eigenvalue of the biharmonic; $E_{m+1} = (\frac{2\pi(m+1)}{L})^4$. Then a sufficient condition for hypothesis (H1) is:

$$\frac{4\pi^2(m+1)^2}{L^2} > \beta + 4\alpha\ [\frac{1}{T}\int_t^{t+T} \max\ (||u^2||^2_{L^\infty}\ ,\ ||v^2||^2_{L^\infty}\ ds]^{\frac{1}{2}} \quad . \tag{4.17}$$

We will further elaborate on (4.17). But we first verify Hypothesis (H2) and (H3) from the generalized Gronwall's Lemma. To verify (H2), notice that (4.14) implies by the Cauchy-Young inequality:

$$\frac{1}{T}\int_t^{t+T} A_m(s)\ ds \geq 2\ \tilde{\rho}_m(t) - 2\ \beta\ \tilde{\rho}_m(t)^{\frac{1}{2}} - \tilde{\rho}_m(t) - \overline{\lim_{t\to\infty}}\ R(u,v)^2 \quad ; \tag{4.18}$$

(H2) is satisfied as soon as

$$\tilde{\rho}_m(t) \geq 4\ \beta^2 \tag{4.19}$$

which is implied by (4.16) and (4.17). To verify (H3), remember that $R(u,v)$ and $||\Delta q_m||$ are uniformly bounded in time (cf., Section 3); moreover, $||p_m(t)|| \to 0$ from the very hypothesis of theorem 4.4.

We now further explicit the remaining sufficient condition (4.17). Using (Lemma 2.1), namely that

$$\bar{u}(t) \equiv \bar{u}(0) \quad ,$$

the continuous injection of $H^1(\Omega)$ into $L^\infty(\Omega)$ can be sharpened as:

$$||u||_{L^\infty} \leq \sqrt{L}\ ||\nabla u||_{L^2} + \bar{u}(0) \quad . \tag{4.20}$$

Then:

$$\left(\frac{1}{T}\int_t^{t+T} \max\ (||u^2||^2_{L^\infty}\ ,\ ||v^2||^2_{L^\infty}\ ds\right)^{\frac{1}{2}}$$

$$\leq \max\ (\overline{\lim_{t\to\infty}}\ ||u||^2_{L^\infty}\ ,\ \overline{\lim_{t\to\infty}}\ ||v||^2_{L^\infty})$$

$$\leq \max\ ((\sqrt{L}\ F(u_0) + \bar{u}(0))^2\ ,\ (\sqrt{L}\ F(v_0) + \bar{v}(0))^2) \quad , \tag{4.21}$$

where we have used Theorem 2.3, i.e., $\overline{\lim_{t\to\infty}}\ ||\nabla u(t)|| \leq F(u_0)$. Then for m and L large enough, (4.17) is equivalent to:

$$\frac{4\pi^2(m+1)^2}{L^2} \sim Ct(\alpha,\beta,u_0,v_0)\ L \quad , \tag{4.22a}$$

$$m + 1 \sim Ct(\alpha,\beta,\zeta_0)\ L^{3/2} \quad , \tag{4.22b}$$

where we have taken both $||\nabla u(0)||$ and $||\nabla v(0)|| < \zeta_0$. □

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